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A Quantitative Version of the Young Test for the Convergence of Conjugate Series

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The classical Young test says that if f is a 2π-periodic function of bounded variation on $[-\pi, \pi]$, then the conjugate series to the Fourier series of f converges at x if and only if the conjugate function $\tilde{f}$ exists at x. Our main goal is to give estimates of the rate of this convergence in terms of the oscillation of $\psi_x(t) := f(x+t) - f(x-t)$ over appropriate subintervals. In particular, we obtain a conjugate version of the well-known Dini–Lipschitz test. As a byproduct, we obtain the rate of convergence in L^1-norm.

1. INTRODUCTION

In this paper, we consider only 2π-periodic functions. Let f be integrable in Lebesgue's sense on $[-\pi, \pi]$. If

$$\frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{1.1}$$

is the Fourier series of f, then the conjugate series to (1.1) is given by

$$\sum_{n=1}^{\infty} (a_n \sin nx - b_n \cos nx). \tag{1.2}$$

Denote by $\tilde{s}_n(f, x)$ the nth partial sum of series (1.2). As it is well known, we have

$$\tilde{s}_n(f, x) = -\frac{1}{\pi} \int_0^{\pi} \psi_x(t)\, \tilde{D}_n(t)\, dt, \qquad n \geqslant 1, \tag{1.3}$$

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where

$$\psi_x(t) := f(x+t) - f(x-t)$$

and

$$\tilde{D}_n(t) := \sum_{k=1}^{n} \sin kt = \frac{\cos t/2 - \cos(n+1/2)\,t}{2 \sin t/2} \tag{1.4}$$

is the conjugate Dirichlet kernel.

The following test for the convergence of (1.2) was given by Young [5]. (See also [6, p. 59].)

THEOREM 0. *If f is of bounded variation on $[-\pi, \pi]$, then a necessary and sufficient condition for the convergence of* (1.2) *at a point x is the existence of the integral*

$$\tilde{f}(x) := \lim_{h \to 0+} \tilde{f}(x, h) := \lim_{h \to 0+} \frac{-1}{\pi} \int_h^\pi \frac{\psi_x(t)}{2 \tan t/2}\, dt, \tag{1.5}$$

which represents then the sum of (1.2).

The function $\tilde{f}$ is said to be conjugate to f. If f is of bounded variation on $[-\pi, \pi]$ and $\tilde{f}$ exists at x, then $\psi_x(t)$ is necessarily continuous at $t=0$.

Our main object is to obtain estimates of the rate of convergence stated in Theorem 0.

2. MAIN RESULTS

We will use the notations

$$I_{kn} := [\theta_{k-1,n}, \theta_{kn}], \qquad \text{where} \quad \theta_{kn} := \frac{k\pi}{n}, \quad k = 0, 1, ..., n; \quad n \geqslant 1.$$

We remind the reader that the oscillation of a bounded function f over an interval I is defined by

$$\operatorname{osc}(f, I) := \sup\{|f(x) - f(y)| : x, y \in I\}.$$

We will prove the following

THEOREM 1. *If f is bounded, then*

$$\left|\tilde{s}_n(f, x) - \tilde{f}\left(x, \frac{\pi}{n}\right)\right| \leqslant \left(1 + \frac{1}{\pi}\right) \sum_{k=1}^{n} \frac{1}{k} \operatorname{osc}(\psi_x, I_{kn}), \qquad n \geqslant 1. \tag{2.1}$$

Hence Theorem 0 follows easily.

We introduce one more notation:

$$J_k := \left[0, \frac{\pi}{k}\right], \qquad k = 1, 2, \ldots .$$

THEOREM 2. *If f is bounded and such that*

$$\sum_{k=1}^{\infty} \frac{1}{k} \operatorname{osc}(\psi_x, J_k) < \infty, \tag{2.2}$$

then $\tilde{f}$ exists at x and

$$|\tilde{s}_n(f, x) - \tilde{f}(x)| \leqslant \left(1 + \frac{1}{\pi}\right) \sum_{k=1}^{n} \frac{1}{k} \operatorname{osc}(\psi_x, I_{kn}) + \frac{1}{\pi} \sum_{k=n}^{\infty} \frac{1}{k} \operatorname{osc}(\psi_x, J_k), \qquad n \geqslant 1. \tag{2.3}$$

In the following, we specialize Theorems 1 and 2 to the particular cases where f is assumed to be (i) continuous or (ii) of bounded variation on $[-\pi, \pi]$.

We remind the reader that the modulus of continuity $\omega(f)$ of a 2π-periodic function f is defined by

$$\omega(f, \delta) := \sup\{|f(x) - f(y)| : |x - y| \leqslant \delta;\ x, y \in \mathbf{R}\}$$

Now, Theorem 1 implies the following

COROLLARY 1. *If f is continuous on $[-\pi, \pi]$, then we have, uniformly in x,*

$$\left|\tilde{s}_n(f, x) - \tilde{f}\left(x, \frac{\pi}{n}\right)\right| \leqslant C\omega\left(f, \frac{\pi}{n}\right) \ln(n+1), \qquad n \geqslant 1. \tag{2.4}$$

Here and in the following, by C we denote a positive constant, not necessarily the same at each occurrence. In general, C may depend on f, but bot on n.

Theorem 2 implies the following result, which is folklore.

COROLLARY 2. *If f is continuous on $[-\pi, \pi]$ and such that*

$$\sum_{k=1}^{\infty} \frac{1}{k} \omega\left(f, \frac{\pi}{k}\right) < \infty, \tag{2.5}$$

then $\tilde{f}$ exists for all x and we have, uniformly in x,

$$|\tilde{s}_n(f, x) - \tilde{f}(x)| \leqslant C\omega\left(f, \frac{\pi}{n}\right) \ln(n+1) + \frac{1}{\pi} \sum_{k=n}^{\infty} \frac{1}{k}\, \omega\left(f, \frac{\pi}{k}\right), \qquad n \geqslant 1. \tag{2.6}$$

The uniform estimates (2.4) and (2.6) may be considered to be the conjugate versions of the well-known Dini–Lipschitz test. (See, e.g., [6, p. 63].)

It is plain that (2.5) is a sufficient condition for the uniform convergence of series (1.2) to the conjugate function $\tilde{f}$. It is easy to see that it is equivalent to the condition

$$\int_0^{\pi} \frac{\omega(f, \delta)}{\delta}\, d\delta < \infty.$$

Now, condition (2.5) is the best possible in the following sense. Let $\omega(\delta)$ be a concave modulus of continuity such that

$$\sum_{k=1}^{\infty} \frac{1}{k}\, \omega\left(\frac{\pi}{k}\right) = \infty. \tag{2.7}$$

According to [4, Lemma 4], for the function

$$f(x) := \sum_{k=2}^{\infty} \left[\omega\left(\frac{\pi}{k}\right) - \frac{k-1}{k}\, \omega\left(\frac{\pi}{k-1}\right)\right] \sin kx$$

we have

$$\omega(f, \delta) \leqslant C\omega(\delta), \qquad \delta \geqslant 0.$$

By definition, the conjugate series is

$$-\sum_{k=2}^{\infty} \left[\omega\left(\frac{\pi}{k}\right) - \frac{k-1}{k}\, \omega\left(\frac{\pi}{k-1}\right)\right] \cos kx =: \tilde{f}(x).$$

Hence

$$\begin{aligned}\tilde{s}_n(f, 0) &= -\sum_{k=2}^{n} \left[\omega\left(\frac{\pi}{k}\right) - \frac{k-1}{k}\, \omega\left(\frac{\pi}{k-1}\right)\right] \\ &= \omega(\pi) - \sum_{k=1}^{n-1} \frac{1}{k+1}\, \omega\left(\frac{\pi}{k}\right) - \omega\left(\frac{\pi}{n}\right)\end{aligned}$$

diverges as $n \to \infty$. Likewise, the conjugate function $\tilde{f}(x)$ does not exist at $x = 0$.

Finally, we prove the following.

THEOREM 3. *If f is of bounded variation on $[-\pi, \pi]$, then*

$$\left|\tilde{s}_n(f, x) - \tilde{f}\left(x, \frac{\pi}{n}\right)\right| \leqslant 9\left(1 + \frac{1}{\pi}\right)\frac{1}{n}\sum_{k=1}^{n} \operatorname{var}(\psi_x, J_k), \qquad n \geqslant 1, \tag{2.8}$$

where $\operatorname{var}(\psi, J)$ *denotes the total variation of the function ψ over the interval J.*

We note that inequality (2.8) was proved by Mazhar and Al-Budaiwi [3] with a smaller constant. The corresponding quantitative version of the classical Dirichlet–Jordan test (see, e.g., [6, p. 57]) was proved by Bojanic [1]. (See also [2].)

3. AUXILIARY RESULTS

LEMMA 1 (See [3]). *We have*

$$\left|\int_x^{\pi} \frac{\cos(n + 1/2)t}{2 \sin t/2}\, dt\right| \leqslant \frac{\pi}{(n + 1/2)\, x}, \qquad 0 < x \leqslant \pi, \quad n \geqslant 0.$$

LEMMA 2. *If ψ is of bounded variation on $[0, \pi]$, then*

$$\sum_{k=1}^{n} \frac{1}{k} \operatorname{osc}(\psi, I_{kn}) \leqslant \frac{9}{n} \sum_{k=1}^{n} \operatorname{var}(\psi, J_k), \qquad n \geqslant 2. \tag{3.1}$$

Proof. By definition,

$$\operatorname{osc}(\psi, I_{kn}) \leqslant \operatorname{var}(\psi, [0, \theta_{kn}]) - \operatorname{var}(\psi, [0, \theta_{k-1,n}]).$$

Hence

$$\sum_{k=1}^{n} \frac{1}{k} \operatorname{osc}(\psi, I_{kn}) \leqslant \frac{1}{n} \operatorname{var}(\psi, [0, \pi]) + \sum_{k=1}^{n-1} \frac{1}{k(k+1)} \operatorname{var}(\psi, [0, \theta_{kn}]). \tag{3.2}$$

Define the nonnegative integer m so $2^m < n \leqslant 2^{m+1}$. By making use of dyadic grouping, we get the following upper estimate:

$$\sum_{k=1}^{n-1} \frac{1}{k(k+1)} \operatorname{var}\left(\psi, \left[0, \frac{k\pi}{n}\right]\right)$$
$$\leqslant \frac{1}{2} \operatorname{var}\left(\psi, \left[0, \frac{\pi}{2^m}\right]\right)$$
$$+ \sum_{j=1}^{m-1} \left(\sum_{k=2^{m-j-1}+1}^{2^{m-j}} \frac{1}{k(k+1)} \right) \operatorname{var}\left(\psi, \left[0, \frac{\pi}{2^j}\right]\right)$$
$$+ \left(\sum_{k=2^{m-1}+1}^{n-1} \frac{1}{k(k+1)} \right) \operatorname{var}(\psi, [0, \pi])$$
$$\leqslant \sum_{j=0}^{m} \frac{1}{2^{m-j-1}} \operatorname{var}\left(\psi, \left[0, \frac{\pi}{2^j}\right]\right). \tag{3.3}$$

Using dyadic grouping again, we can estimate from below the sum on the right-hand side of (3.1) as follows:

$$\frac{1}{n} \sum_{k=1}^{n} \operatorname{var}\left(\psi, \left[0, \frac{\pi}{k}\right]\right) \geqslant \frac{1}{2^{m+1}} \left\{ \operatorname{var}(\psi, [0, \pi]) + \sum_{j=1}^{m} 2^{j-1} \operatorname{var}\left(\psi, \left[0, \frac{\pi}{2^j}\right]\right) \right\}$$
$$\geqslant \sum_{j=0}^{m} \frac{1}{2^{m-j+2}} \operatorname{var}\left(\psi, \left[0, \frac{\pi}{2^j}\right]\right). \tag{3.4}$$

Combining (3.2)–(3.4) yields

$$\sum_{k=1}^{n} \frac{1}{k} \operatorname{osc}(\psi, I_{kn}) \leqslant \frac{1}{n} \operatorname{var}(\psi, [0, \pi]) + \frac{8}{n} \sum_{k=1}^{n} \operatorname{var}(\psi, J_k).$$

This proves (3.1).

We note that in the case $n = 2^m$ with some nonnegative integer m, the above proof provides a smaller constant, namely 5 instead of 9.

4. PROOFS

Proof of Theorem 1. By (1.3)–(1.5),

$$\tilde{s}_n(f, x) - \tilde{f}\left(x, \frac{\pi}{n}\right)$$
$$= -\frac{1}{\pi} \int_0^{\pi/n} \psi_x(t)\, \tilde{D}_n(t)\, dt + \frac{1}{\pi} \int_{\pi/n}^{\pi} \psi_x(t) \frac{\cos(n+1/2)\,t}{2 \sin t/2}\, dt$$

$$= -\frac{1}{\pi}\int_0^{\pi/n} \psi_x(t)\,\tilde{D}_n(t)\,dt + \frac{1}{\pi}\sum_{k=2}^{n}\int_{I_{kn}} [\psi_x(t)-\psi_x(\theta_{k-1,n})]$$

$$\times \frac{\cos(n+1/2)\,t}{2\sin t/2}\,dt$$

$$+\frac{1}{\pi}\sum_{k=2}^{n}\psi_x(\theta_{k-1,n})\int_{I_{kn}}\frac{\cos(n+1/2)\,t}{2\sin t/2}\,dt =: A_n + B_n + C_n, \qquad (4.1)$$

say. Since

$$|\tilde{D}_n(t)| \leqslant n \qquad \text{and} \qquad \psi_x(0)=0,$$

we have

$$|A_n| \leqslant \operatorname{osc}(\psi_x, I_{1n}). \qquad (4.2)$$

Making use of the obvious estimate

$$\left|\frac{\cos(n+1/2)\,t}{2\sin t/2}\right| \leqslant \frac{\pi}{2t}, \qquad 0 < t \leqslant \pi,$$

we find

$$|B_n| \leqslant \frac{1}{\pi}\sum_{k=2}^{n}\operatorname{osc}(\psi_x, I_{kn})\int_{I_{kn}}\frac{\pi}{2t}\,dt$$

$$\leqslant \frac{1}{2}\sum_{k=2}^{n}\frac{1}{k-1}\operatorname{osc}(\psi_x, I_{kn})$$

$$\leqslant \sum_{k=2}^{n}\frac{1}{k}\operatorname{osc}(\psi_x, I_{kn}). \qquad (4.3)$$

Setting

$$R_{kn} := \int_{\theta_{kn}}^{\pi}\frac{\cos(n+1/2)\,t}{2\sin t/2}\,dt, \qquad k=1,2,\ldots,n,$$

by Lemma 1,

$$|R_{kn}| \leqslant \frac{\pi}{(n+1/2)\,\theta_{kn}} < \frac{1}{k}, \qquad R_{nn}=0.$$

By summation by parts,

$$C_n := \frac{1}{\pi} \sum_{k=2}^{n} \psi_x(\theta_{k-1,n})(R_{k-1,n} - R_{kn})$$
$$= \frac{1}{\pi} \sum_{k=1}^{n-1} [\psi_x(\theta_{kn}) - \psi_x(\theta_{k-1,n})] R_{kn},$$

whence

$$|C_n| \leqslant \frac{1}{\pi} \sum_{k=1}^{n-1} \frac{1}{k} \operatorname{osc}(\psi_x, I_{kn}). \tag{4.4}$$

Combining (4.1)-(4.4) gives (2.1).

Proof of Theorem 2. If

$$\frac{\pi}{n+1} < h \leqslant \frac{\pi}{n}, \qquad n \geqslant 1,$$

then

$$\left|\tilde{f}(x, h) - \tilde{f}\left(x, \frac{\pi}{n+1}\right)\right| \leqslant \frac{1}{\pi} \int_{\pi/(n+1)}^{\pi/n} \frac{|\psi_x(t)|}{2 \tan t/2} \, dt$$
$$\leqslant \frac{1}{n\pi} \operatorname{osc}(\psi_x, J_n).$$

Analogously, for such h we have

$$|\tilde{f}(x, h) - \tilde{f}(x)| \leqslant \left|\tilde{f}(x, h) - \tilde{f}\left(x, \frac{\pi}{n+1}\right)\right|$$
$$+ \sum_{k=n+1}^{\infty} \left|\tilde{f}\left(x, \frac{\pi}{k}\right) - \tilde{f}\left(x, \frac{\pi}{n+1}\right)\right|$$
$$\leqslant \frac{1}{\pi} \sum_{k=n}^{\infty} \frac{1}{k} \operatorname{osc}(\psi_x, J_k).$$

In particular, this is true when $h := \pi/n$.

Now, Theorem 2 follows from Theorem 1 and (2.2).

Proof of Theorem 3. Inequality (2.8) is an immediate consequence of Theorem 1 and Lemma 2.

5. Covergence in L^1-Norm

Imitating the proofs of Theorems 1 and 2, we may obtain the following quantitative versions of the Young test in L^1-norm.

Theorem 1*. *If f is integrable on $[-\pi, \pi]$, then*

$$\int_{-\pi}^{\pi} \left|\tilde{s}_n(f, x) - \tilde{f}\left(x, \frac{\pi}{n}\right)\right| dx \leqslant \left(1 + \frac{1}{\pi}\right) \sum_{k=1}^{n} \frac{1}{k} \Omega(\psi, I_{kn}), \qquad n \geqslant 1,$$

where

$$\Omega(\psi, I) := \sup \left\{\int_{-\pi}^{\pi} |\psi_x(t) - \psi_x(t')| \, dx : t, t' \in I\right\}.$$

Theorem 2*. *If f is integrable on $[-\pi, \pi]$ and such that*

$$\sum_{k=1}^{\infty} \frac{1}{k} \Omega(\psi, J_k) < \infty,$$

then $\tilde{f}$ is also Lebesgue integable and

$$\int_{-\pi}^{\pi} |\tilde{s}_n(f, x) - \tilde{f}(x)| \, dx \leqslant \left(1 + \frac{1}{\pi}\right) \sum_{k=1}^{n} \frac{1}{k} \Omega(\psi, I_{kn}) + \frac{1}{\pi} \sum_{k=n}^{\infty} \frac{1}{k} \Omega(\psi, J_k), \qquad n \geqslant 1.$$

It is plain that

$$\Omega(\psi, I_{kn}) \leqslant 2\omega_1\left(f, \frac{\pi}{n}\right),$$

where

$$\omega_1(f, \delta) := \sup \left\{\int_{-\pi}^{\pi} |f(x+t) - f(x)| \, dx : |t| \leqslant \delta\right\}$$

is the integral modulus of continuity of f.

Now, the conjugate versions of the Dini–Lipschitz test in L^1-norm reads as follows.

Corollary 1*. *If f is integrable on $[-\pi, \pi]$, then*

$$\int_{-\pi}^{\pi} \left|\tilde{s}_n(f, x) - \tilde{f}\left(x, \frac{\pi}{n}\right)\right| dx \leqslant C\omega_1\left(f, \frac{\pi}{n}\right) \ln(n+1), \qquad n \geqslant 1.$$

COROLLARY 2*. *If f is integrable on $[-\pi, \pi]$ and such that*

$$\sum_{k=1}^{\infty} \frac{1}{k} \omega_1\left(f, \frac{\pi}{k}\right) < \infty, \tag{5.1}$$

then $\tilde{f}$ is also Lebesgue integrable and

$$\int_{-\pi}^{\pi} |\tilde{s}_n(f, x) - \tilde{f}(x)|\, dx \leqslant C\omega_1\left(f, \frac{\pi}{n}\right) \ln(n+1) + \frac{1}{\pi} \sum_{k=n}^{\infty} \frac{1}{k} \omega_1\left(f, \frac{\pi}{k}\right), \qquad n \geqslant 1.$$

Clearly, (5.1) is equivalent to the condition

$$\int_0^{\pi} \frac{\omega_1(f, \delta)}{\delta}\, d\delta < \infty.$$

Problem. Is condition (5.1) the best possible in the following sense: Given a concave modulus of continuity $\omega(\delta)$ such that condition (2.7) is satisfied, does there exist an integrable function f such that

$$\omega_1(f, \delta) \leqslant C\omega(\delta), \qquad \delta \geqslant 0,$$

but $\tilde{f}$ is not Lebesgue integrable, or at least, the conjugate series (1.2) does not converge to $\tilde{f}$ in the L-norm?

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